

# EMBEDDINGS OF LOCALLY COMPACT HYPERBOLIC GROUPS INTO $L_p$ -SPACES

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**ABSTRACT.** In the last years, there has been a large amount of research on embeddability properties of finitely generated hyperbolic groups. In this paper, we elaborate on the more general class of locally compact hyperbolic groups. Viewing  $SO(n, 1)$  as a locally compact hyperbolic group and by proving that the equivariant  $L_p$ -compression of a locally compact compactly generated group is minimal for  $p = 2$ , we calculate all equivariant  $L_p$ -compressions of  $SO(n, 1)$ . Next, we show that although there are locally compact, non-discrete hyperbolic groups  $G$  with Kazhdan property  $(T)$ , it is true that any locally compact hyperbolic group admits a proper affine isometric action on an  $L_p$ -space for  $p$  larger than the Ahlfors regular conformal dimension of  $\partial G$ . This answers a question asked by Yves de Cornulier. In the finitely generated case, this result follows from the work of M. Bourdon (see also the work of G. Yu and B. Nica). Using our result and applying it to  $Sp(n, 1)$  and  $F_4^{-20}$  in particular leads to a new proof of the fact that  $Sp(n, 1)$  (and  $F_4^{-20}$ ) admit proper affine isometric actions on  $L_p$ -spaces for  $p > 4n + 2$  (and  $p > 22$  respectively).

## 1. INTRODUCTION

**1.1. Locally compact hyperbolic groups.** The common convention when dealing with hyperbolic groups is that such groups are finitely generated and equipped with the word length metric relative to a finite generating subset. On the one hand, this leads to a very interesting theory with very strong results, but on the other hand, this class misses interesting groups such as  $SO(n, 1)$ ,  $SU(n, 1)$ ,  $Sp(n, 1)$  that contain hyperbolic uniform lattices. Gromov's work [22] already contained ideas which encompass locally compact hyperbolic groups. Following [17] we define a locally compact group  $G$  to be **hyperbolic** if it is compactly generated and word-hyperbolic with respect to the word length metric relative to some compact generating subset. This is equivalent to the group acting continuously, properly, cocompactly and isometrically on a proper hyperbolic geodesic metric space  $X$  (see Corollary 2.6 of [12]). The space  $X$  is determined up to quasi-isometry and so one can unambiguously define the hyperbolic boundary  $\partial G$  of  $G$  as the hyperbolic boundary of  $X$ . Some examples of locally compact hyperbolic groups are  $SO(n, 1)$ ,  $SU(n, 1)$ ,  $Sp(n, 1)$ ,  $F_4^{-20}$  and the class of groups of the form  $H \rtimes_{\alpha} \mathbb{Z}$ ,  $H \rtimes_{\alpha} \mathbb{R}$  where  $\alpha(1)$  is such that there is some compact set  $V \subseteq H$  such that  $\forall x \in H, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 : \alpha(n)(x) \in V$  (see [12]).

There are many, also non-trivial, results that generalize to the locally compact setting. For example, in [13], the authors generalize Bowditch's topological characterization of discrete hyperbolic groups to the locally compact setting. This for starters has a nice application in the study of sharply- $n$ -transitive actions on compact sets. All groups acting sharply- $n$ -transitively on compact spaces  $M$  are completely classified, except for  $n = 3$ . The authors show that  $\sigma$ -compact groups acting sharply-3-transitively on a compact space  $M$  are necessarily locally compact hyperbolic, that  $M$  is homeomorphic to the boundary of the group and that the action on  $M$  coincides, via this homeomorphism, with the natural action of a hyperbolic group on its boundary.

On the other hand, as the discrete hyperbolic groups merely constitute a special case of the more general locally compact setting, one can not simply hope to extend all results from the discrete to the locally compact context. For example, from the discrete case, the intuition has grown that hyperbolicity and amenability are somehow incompatible: it is known that non-elementary

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finitely generated hyperbolic groups contain a free non-abelian subgroup  $F_2$  and are thus non-amenable. On the other hand, in [17], the authors prove the counter-intuitive fact that there *do* exist *amenable* non-elementary locally compact hyperbolic groups! There are plenty of other differences as well: for example, it is possible for locally compact hyperbolic groups to act transitively on their boundary, even if the boundary is infinite. In fact, such groups are studied in [12], [13]. On the other hand, non-elementary discrete hyperbolic groups are countable, and so cannot act transitively on their uncountable boundary. The list of differences is endless.

The behaviour of groups with respect to embeddings into  $L_p$ -spaces ( $p \geq 1$ ) is directly related to properties such as the Haagerup property, property  $A$ , property  $T$ , coarse embeddability and hence it is related to important conjectures such as the Novikov and Baum-Connes conjecture. Recently, a lot of work was done investigating the equivariant and non-equivariant embeddability behaviour of discrete hyperbolic groups into  $L_p$ -spaces ( $p \geq 1$ ) (see e.g. [39], [9], [41], [4], [34] and others). For example, it was shown in [41], that although there are finitely generated hyperbolic groups with property  $(T)$  (which thus do not admit a proper affine isometric action on an  $L_2$ -space), each discrete hyperbolic group admits a proper affine isometric action on an  $L_p$ -space for  $p$  sufficiently large! One of the things that we will show, is that this result persists in the locally compact hyperbolic setting. Moreover, we will be more specific and quantify *how proper* the action is, as this is related to group theoretic properties such as amenability and exactness. We give the necessary definitions before formulating our results.

## 1.2. Equivariant and non-equivariant $L_p$ -compression.

**Definition 1.1** (see [21]). *Fix  $p \geq 1$ . A compactly generated group  $G$  is **coarsely embeddable into an  $L_p$ -space**, if there exists a measure space  $(\Omega, \mu)$ , a non-decreasing function  $\rho_- : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\lim_{t \rightarrow \infty} \rho_-(t) = +\infty$ , a constant  $C > 0$  and a map  $f : G \rightarrow L_p(\Omega, \mu)$ , such that*

$$\rho_-(d(g, h)) \leq \|f(g) - f(h)\|_p \leq Cd(g, h) \quad \forall g, h \in G,$$

*where  $d$  is the word length metric relative to a compact generating subset. The map  $f$  is called a **coarse embedding** of  $G$  into  $L_p(\Omega, \mu)$  and the map  $\rho_-$  is called a **compression function** for  $f$ .*

**Definition 1.2.** *Let  $G$  be a group and  $(\Omega, \mu)$  a measure space. Fix  $p \geq 1$ . A map  $f : G \rightarrow L_p(\Omega, \mu)$  is called  **$G$ -equivariant**, if there is an affine isometric action  $\alpha$  of  $G$  on  $L_p(\Omega, \mu)$  such that  $\forall g, h \in G : f(gh) = \alpha(g)(f(h))$ . A locally compact second countable group  $G$  admitting an equivariant coarse embedding into a Hilbert space is said to satisfy the **Haagerup property**.*

The Haagerup property is a subject of intense study [15] and is known to imply the strong Baum-Connes conjecture. In 2004, Guentner and Kaminker introduce two numerical invariants [24]. The first quantifies how well a group satisfies the Haagerup property, i.e. how well it embeds coarsely and equivariantly into an  $L_2$ -space. The other quantifies how well the group embeds coarsely, but not necessarily equivariantly, into an  $L_2$ -space. The latter invariant links coarse embeddability to the well-studied notion of quasi-isometric embeddability [19].

**Definition 1.3.** *Fix  $p \geq 1$ . Given a compactly generated locally compact group  $G$  and a measure space  $(\Omega, \mu)$ , the  **$L_p$ -compression**  $R(f)$  of a coarse embedding  $f : G \rightarrow L_p(\Omega, \mu)$  is defined as the supremum of  $r \in [0, 1]$  such that*

$$\exists C, D > 0, \forall g, h \in G : \frac{1}{C}d(g, h)^r - D \leq \|f(g) - f(h)\| \leq Cd(g, h).$$

*The **equivariant  $L_p$ -compression**  $\alpha_p^*(G)$  of  $G$  is defined as the supremum of  $R(f)$  taken over all  $G$ -equivariant coarse embeddings of  $G$  into all possible  $L_p$ -spaces. Taking the supremum of  $R(f)$  over all, also non-equivariant, coarse embeddings, leads to the (non-equivariant)  **$L_p$ -compression**  $\alpha_p(G)$  of  $G$ . One can show that both of these definitions do not depend on the chosen compact generating neighbourhood.*

Equivariant and non-equivariant compression are related to interesting group theoretic properties. Indeed, based on a remark by M. Gromov, it was shown that the equivariant and non-equivariant  $L_2$ -compression are equal for amenable groups (see [19]). Moreover, if the equivariant

$L_2$ -compression of a compactly generated group is  $> \frac{1}{2}$  then the group is amenable [19]. This provides a partial converse to the statement that amenable groups satisfy the Haagerup property. There is a similar statement for the equivariant  $L_p$ -compression for  $p \neq 2$ : in [33], the authors show that a finitely generated group is amenable if its equivariant  $L_p$ -compression is strictly greater than  $\frac{1}{\min(2,p)}$ . In the non-equivariant setting, we have the result that finitely generated groups with non-equivariant compression  $> 1/2$  satisfy non-equivariant amenability, i.e. property A (which is equivalent to exactness of the reduced  $C^*$ -algebra) [24]. Moreover, for finitely generated groups, the non-equivariant Haagerup property, i.e. coarse embeddability into a Hilbert space, implies the Novikov conjecture. This result was suggested by Gromov in [23], but he did not give a proof. Around 2000, in a breakthrough article, Yu showed that such groups satisfy the coarse Baum-Connes conjecture, which is related to the Novikov conjecture [42]. Together with Skandalis and Tu, the Novikov conjecture for finitely generated coarsely embeddable groups was proven in [38]. Later, in [28], the authors prove the same result for embeddings into uniformly convex Banach spaces and this is one of the motivations to study embeddings into  $L_p$ -spaces for  $p \neq 2$ .

**1.3. Results.** In [35], P. Nowak characterizes the Haagerup property geometrically by proving that a locally compact second countable group is Haagerup if and only if it admits a proper, affine isometric action on some  $L_p([0, 1])$  for  $1 < p < 2$ . This work was motivated by a question asked by A. Valette (see Section 7.4.2 in [15]). In [14], the authors show that any group  $G$  with the Haagerup property also admits a proper affine isometric action on any  $L_p$ -space  $p \geq 1$ . By analysing their methods, we find moreover the lower bound  $\alpha_p^*(G) \geq \alpha_2^*(G)/p$  for all  $p \geq 1$ . Naor and Peres prove the astonishing fact for finitely generated groups that the equivariant  $L_2$ -compression is actually minimal among all the other equivariant  $L_p$ -compressions ( $p \geq 1$ ) (Lemma 2.3 in [33]). We prove this for the class of locally compact compactly generated groups.

**Theorem 1.4** (See Theorem 4.2). *For every locally compact, compactly generated group  $G$ , we have  $\alpha_p^*(G) \geq \alpha_2^*(G)$  for all  $p \geq 1$ .*

This observation leads to the following two corollaries.

**Corollary 1.5.** *For each  $n$ , we have  $\alpha_p^*(SU(n, 1)) = 1/2$  when  $p \geq 2$ .*

We can furthermore calculate all the equivariant  $L_p$ -compressions for  $G = SO(n, 1)$ .

**Corollary 1.6.** *For each  $p, n \geq 1$ , we have  $\alpha_p^*(SO(n, 1)) = \max(1/2, 1/p)$ .*

We refer to Corollaries 4.5 and 4.6 in the text for the proofs.

Next, we study locally compact hyperbolic groups. Given two functions  $\rho_1, \rho_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , we write  $\rho_1 \preceq \rho_2$  if there are constants  $C, M > 0$  such that  $\rho_1(t) \leq C\rho_2(t)$  for all  $t \geq M$ . If  $\rho_1 \preceq \rho_2$  and  $\rho_2 \preceq \rho_1$ , then we say that  $\rho_1$  and  $\rho_2$  are *equivalent*. In Theorem 3 and 4 of [39], R. Tessera describes the embeddability behaviour of infinite (binary) trees into  $L_p$ -spaces. Using [10], we show that a locally compact hyperbolic group embeds into a finite product of binary trees. We obtain the following result.

**Theorem 1.7** (See Theorem 3.1). *Let  $G$  be a locally compact hyperbolic group and choose  $p \geq 1$ . If  $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfies*

$$\int_1^\infty \left(\frac{\rho(t)}{t}\right)^p \frac{dt}{t} < \infty,$$

*then there exists a coarse embedding of  $G$  into an  $L_p$ -space with compression function equivalent to  $\rho$ . On the other hand, if there exists a coarse embedding of  $G$  into an  $L_p$ -space with compression function  $\rho$ , then  $\rho$  must satisfy*

$$\int_1^\infty \left(\frac{\rho(t)}{t}\right)^{\max(2,p)} \frac{dt}{t} < \infty.$$

An immediate consequence of Theorem 1.7 is that  $\alpha_p(G) = 1$  for every  $p \geq 1$  and any locally compact hyperbolic group. This result was known for discrete hyperbolic groups and connected Lie groups [39]. Not much was known for non-discrete hyperbolic groups which are not connected

Lie groups, e.g. for totally disconnected non-discrete hyperbolic groups. As a corollary of our Theorems 1.4 and 1.7, we will also obtain the following.

**Corollary 1.8** (See Corollary 4.4). *Any amenable locally compact hyperbolic group  $G$  satisfies  $\alpha_p^*(G) = 1$  for every  $p \geq 1$ .*

Finally, we elaborate on Yu's result that finitely generated hyperbolic groups admit proper affine isometric actions, with compression  $\geq 1/p$ , on an  $L_p$ -space for  $p$  sufficiently large [41] (this result also follows independently from both the work of B. Nica [34] and the work of M. Bourdon [4]). Bourdon shows moreover that  $p$  is *sufficiently large* when it is strictly larger than the Ahlfors regular conformal dimension of the hyperbolic boundary of the group (see [4]). Yves de Cornulier asked whether a similar result would be valid in the locally compact hyperbolic setting. We answer this question affirmatively. Note that we are allowed to exclude the amenable case by Corollary 1.8.

**Theorem 1.9** (See Theorem 6.8). *Let  $G$  be a non-amenable locally compact non-elementary hyperbolic group and denote the Ahlfors regular conformal dimension of  $\partial G$  by  $Q$ . Then for each  $p > Q$ , there is a metrically proper affine isometric action of  $G$  on a finite  $l_p$ -direct sum of copies of  $L_p(G)$ . The linear part of the action is the finite direct sum of the natural translation actions. One can choose the corresponding 1-cocycle to have compression function  $\succeq t^{1/p}$ . In particular, we thus have  $\alpha_p^*(G) \geq 1/p$ .*

**Remark 1.10.** *By combining property (T) and a result by Fisher-Margulis, there exists  $\epsilon(G) > 0$  such that  $\alpha_p^*(G) = 0$  for  $1 \leq p \leq 2 + \epsilon(G)$ . On the other hand, note that our result shows that  $\epsilon(G)$  can not be arbitrarily large when  $G$  is locally compact hyperbolic.*

**Remark 1.11.** *Bogdan Nica [34] proved that finitely generated hyperbolic groups admit a proper affine isometric action on  $l^p(\partial G \times \partial G)$  for  $p$  sufficiently large. The advantage of this result is that the  $L_p$ -space on which the group acts, is explicitly known: it is  $l^p(\partial G \times \partial G)$ . The down side is that he does not obtain a nice lower bound on the minimal  $p$  for which this is true. Bogdan Nica informed us that it should be possible to generalize his proof to the locally compact hyperbolic groups that contain a finitely generated hyperbolic and cocompact subgroup.*

Pierre Pansu showed that the Ahlfors regular conformal dimension of the boundary of  $G = Sp(n, 1), F_4^{-20}$  is known to be  $4n + 2$  and 22 respectively [36]. Based on the observation that  $G$  is hyperbolic, our methods thus also provide a new proof for the following.

**Corollary 1.12.** *The group  $G = F_4^{-20}$  for  $p > 22$  and the groups  $G = Sp(n, 1)$  for  $p > 4n + 2$  admit proper affine isometric actions on an  $L_p$ -space such that the associated 1-cocycle has compression at least  $1/p$ .*

In [18], the authors' main result is that  $Sp(n, 1)$  (and  $F_4^{-20}$ ) admit a proper affine isometric action on  $L_p(Sp(n, 1))$  ( $L_p(F_4^{-20})$ ) respectively for  $p > 4n + 2$  ( $p > 22$  respectively).

## 2. PRELIMINARIES

**2.1. Visual metrics and shadows.** Recall that a compactly generated locally compact group is called **elementary** if either it has no ends (i.e. it is compact) or it has 2 ends (i.e. it acts properly and cocompactly by isometries on the Euclidean line). For our purposes, the elementary case is trivial and we make the following convention.

**Convention 2.1.** *Throughout this section,  $G$  is a non-elementary, locally compact, hyperbolic group, equipped with the word length metric relative to a compact generating subset. Throughout this paper,  $(X, d)$  will always denote a proper (i.e. closed balls are compact) hyperbolic geodesic metric space on which  $G$  acts properly and cocompactly by isometries. We denote the hyperbolicity constant of  $X$  by  $\delta$ .*

Similarly to the finitely generated case, one defines the hyperbolic boundary  $\partial G$  of  $G$  as the hyperbolic boundary  $\partial X$  of  $(X, d)$ , i.e. as the set of equivalence classes of geodesic rays in  $X$  where two rays are said to be **equivalent** if they lie at bounded Hausdorff distance from each

other. Note that, as  $G$  acts by isometries on  $X$ , and thus maps geodesic rays to geodesic rays, one obtains a natural  $G$ -action on  $\partial X = \partial G$ . It is well known that the boundary of  $X$ , and thus of  $G$ , is a compact, perfect set, metrizable by a *visual metric*.

**Definition 2.2.** A metric  $\rho$  on the hyperbolic boundary of a proper hyperbolic geodesic metric space  $(X, d)$  is called **visual** if

- (1)  $\rho$  induces the canonical boundary topology on  $\partial X$
- (2) there exist  $x_0 \in X, C > 0$  and  $a > 1$  such that for any two distinct  $\xi, \xi' \in \partial X$  and for any bi-infinite geodesic  $\gamma$  in  $X$  connecting  $\xi$  to  $\xi'$  and any  $y \in \gamma$  with  $d(x_0, y) = d(x_0, \gamma)$  we have

$$\frac{1}{C}a^{-d(x_0, y)} \leq \rho(\xi, \xi') \leq Ca^{-d(x_0, y)}.$$

We sometimes denote  $\rho = d_a$  (or  $d_{a,C}$ ) if we need to emphasize the value of  $a$  (and/or  $C$ ).

It is a standard fact that two visual metrics  $d_a, d_b$  on  $\partial X$  are Hölder equivalent [27]. Stronger even, there exist  $C > 0$  and  $\alpha > 0$  such that for any  $\xi, \xi' \in \partial X$ :

$$d_b(\xi, \xi')^\alpha / C \leq d_a(\xi, \xi') \leq Cd_b(\xi, \xi')^\alpha.$$

Given a ball in  $X$ , say of radius  $R$  and centre  $x$ , we define its shadow by

$$\mathcal{S}(x, R) = \{\xi \in \partial X \mid \exists \text{ a geodesic ray } r \in \xi \text{ starting in } x_0 \text{ such that } r \cap B(x, R) \neq \emptyset\}.$$

We state the following lemma for reference sake.

**Lemma 2.3.** *There exists  $R > 0$  such that  $\mathcal{S}(x, R) \neq \emptyset$  for every  $x \in X$ .*

*Proof.* To see that such an  $R$  exists, choose two distinct points in  $\partial X$  and take a bi-infinite geodesic in  $X$  connecting them. Let  $y$  be any point on this geodesic. By cocompactness, there exists some  $R' > 0$  and a closed ball  $y \in C \subset X$  of diameter  $R'$  such that  $GC = X$ . So, if  $x \in X$  is any element, then we can find  $\gamma \in G$  mapping  $y$  into a ball of radius  $R'$  around  $x$ . The image of a bi-infinite geodesic under  $\gamma$  is again a bi-infinite geodesic  $(\xi, \xi')$ . Using the fact that ideal triangles are  $20\delta$ -thin, we see that  $[1, \xi)$  or  $[1, \xi')$  passes through the ball of radius  $R := R' + 20\delta$  around  $x$ . So,  $\xi$  or  $\xi'$  lies in  $\mathcal{S}(x, R)$ .  $\square$

**2.2. Q.I.-embedded free subgroups of locally compact hyperbolic groups.** Given a locally compact hyperbolic group  $G$ , it follows from [22] that its boundary is either empty, finite or uncountable. In the latter case, we say that  $G$  is **non-elementary** hyperbolic. Following [22], this case has two important subcases:

- (1) The action of  $G$  on its boundary is called **focal** if and only if  $G$  fixes a point  $\xi \in \partial X$ . In this case,  $\{\xi\}$  is the unique finite orbit of the action.
- (2) The action of  $G$  on its boundary is called **of general type** if it has no finite orbit.

Lemma 5.2 and 5.3 in [12] show that a non-elementary locally compact hyperbolic group is amenable if and only if the action on its boundary is focal. An application of the ping-pong lemma (see [22], 8.2.E, 8.2.F) yields the following result.

**Lemma 2.4** (Lemma 3.3 in [12]). *Every non-elementary locally compact hyperbolic group  $G$  contains a quasi-isometrically embedded copy of a free (non-abelian) subsemigroup on two generators. If the action of  $G$  on its boundary is of general type, then the subsemigroup can be taken to be the free group on two generators  $F_2$ .*

The focal case has been characterized and turns out to be quite restricted (see Theorem A in [12]). Using the standard fact that  $\alpha_p^*(F_2) = \max(1/2, 1/p)$  for  $p \geq 1$  (see [33]), we immediately obtain the following corollary.

**Corollary 2.5.** *The equivariant  $L_p$ -compression ( $p \geq 1$ ) of a locally compact hyperbolic group  $G$  of general type is bounded from above by  $\alpha_p^*(F_2) = \max(1/2, 1/p)$ . In particular,  $\alpha_p^*(SO(n, 1)) \leq \max(1/2, 1/p)$  and  $\alpha_p^*(SU(n, 1)) \leq \max(1/2, 1/p)$  for all  $n \in \mathbb{N}_0$  and  $p \geq 1$ .*

### 3. NON-EQUIVARIANT EMBEDDABILITY OF LOCALLY COMPACT HYPERBOLIC GROUPS

In [39] (see Theorems 3 and 4), R. Tessera (see [39]) showed that for any  $p \geq 1$  and any function  $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfying

$$(1) \quad \int_1^\infty \left(\frac{\rho(t)}{t}\right)^p \frac{dt}{t} < \infty,$$

there exists a coarse embedding of any simplicial tree into  $l^p(\mathbb{Z})$  with compression function  $\rho$ . Moreover, if there exists a coarse embedding of the infinite binary tree into an  $L_p$ -space with compression function  $\rho$ , then  $\rho$  must satisfy

$$(2) \quad \int_1^\infty \left(\frac{\rho(t)}{t}\right)^{\max(2,p)} \frac{dt}{t} < \infty.$$

The exponent  $\max(2, p)$  stems from the fact that Tessera generalizes a result from Bourgain for  $q$ -uniformly convex Banach spaces (see [7]) and the fact that  $L_p$ -spaces are  $\max(2, p)$ -uniformly convex.

In [39], Tessera shows among other things that discrete hyperbolic groups, connected Lie groups and polycyclic groups admit a coarse embedding into an  $L_p$ -space with compression function  $\rho$  if  $\rho$  satisfies Equation (1). The same is thus true for connected hyperbolic Lie groups. Nothing was known about locally compact hyperbolic groups which are not connected Lie.

The main observation in this section, is that one can extend Tessera's result to the general locally compact hyperbolic setting. First, by Lemma 2.4, we have that any locally compact hyperbolic group  $G$  contains a quasi-isometrically embedded copy of the infinite binary tree. So, using Equation (2), we see that the compression function  $\rho$  of any coarse embedding of  $G$  into an  $L_p$ -space must satisfy

$$\int_1^\infty \left(\frac{\rho(t)}{t}\right)^{\max(2,p)} \frac{dt}{t} < \infty.$$

On the other hand, we mention that a compact metric space  $(Z, d)$  is called **doubling** if there exists a constant  $C \geq 1$  such that for any  $r > 0$ , each open ball of radius  $r > 0$  can be covered by at most  $C$  open balls of radius  $r/2$ . By verifying the conditions in [5], bottom of page 2, we see that the boundary of a locally compact hyperbolic group, equipped with a visual metric, is doubling. By our Lemma 2.3, all of the conditions of Theorem 1.2 in [10] are then satisfied. This result then implies that a locally compact hyperbolic group embeds quasi-isometrically into a finite product of binary trees. Changing this product by an  $l^p$ -direct sum for any  $p \geq 1$ , does not change the quasi-isometry type of our finite product of trees. So, we can apply the above mentioned results of Tessera (see Equation (1)) to conclude the following.

**Theorem 3.1.** *Let  $G$  be a locally compact hyperbolic group and fix  $p \geq 1$ . If  $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfies*

$$\int_1^\infty \left(\frac{\rho(t)}{t}\right)^p \frac{dt}{t} < \infty,$$

*then there exists a coarse embedding of  $G$  into  $l^p(\mathbb{Z})$  with compression function  $\rho$ . Conversely, if there exists a coarse embedding of  $G$  into an  $L_p$ -space with compression function  $\rho$ , then  $\rho$  satisfies*

$$\int_1^\infty \left(\frac{\rho(t)}{t}\right)^{\max(2,p)} \frac{dt}{t} < \infty.$$

**Remark 3.2.** *The fact that locally compact hyperbolic groups embed into a finite product of (not necessarily infinite binary) trees, could have also been deduced from a result of Schramm and Bonk as follows. We say that a metric space  $(X, d)$  has **bounded growth at some scale** if  $\exists r, R \in \mathbb{R}$  with  $R > r > 0$  and there exists  $N \in \mathbb{N}$  such that every  $R$ -ball in  $X$  is covered by  $N$  balls of radius  $r$ . One can check easily that this property is a quasi-isometric invariant. So, the proper geodesic metric space  $X$  associated to a locally compact hyperbolic group has bounded growth at some scale. Now, following [9], note that Theorem 1.1 in [2] states that a Gromov hyperbolic geodesic metric space with bounded growth at some scale, is quasi-isometric to a convex subset of hyperbolic  $n$ -space. Hyperbolic space in turn embeds quasi-isometrically into a finite product of trees equipped*

with the  $l_1$ -metric [11]. Given any  $p \geq 1$ , this product is quasi-isometric to the same product equipped with the  $l_p$ -metric. Using Equation (1), we can then conclude similarly as before.

#### 4. THE (EQUIVARIANT) $L_2$ -COMPRESSION OF LOCALLY COMPACT COMPACTLY GENERATED GROUPS IS MINIMAL

It is known that  $L_2$  embeds isometrically into  $L_p$  for any  $p$  (see [1], page 189) so the non-equivariant  $L_2$ -compression is minimal among all  $L_p$ -compressions. The equivariant case is harder to solve.

**Remark 4.1.** A first approach to tackle the equivariant case could be to quantify the results in [14]. This leads to the lower bound  $\alpha_p^*(G) \geq \alpha_2^*(G)/2$  for all  $p \geq 1$  and all compactly generated locally compact groups  $G$ . We give a brief overview of how we obtained this bound. For underlying definitions, we refer the reader to [14].

Given a proper affine isometric action of a group  $G$  on a Hilbert space  $\mathcal{H}$ , let  $b : G \rightarrow \mathcal{H}$  be the orbit map of  $0 \in \mathcal{H}$ . Assume that there is some  $\alpha > 0$  such that  $\|b(g)\| \geq \frac{1}{C}l(g)^\alpha$  for all  $g \in G$  and where  $l$  is the word length of  $G$  with respect to a compact generating subset. It is well known that  $\psi : (g, g') \mapsto \|b(g) - b(g')\|^2$  is a so called conditionally negative definite kernel. From Corollary 6.18 and Lemma 6.15 in [14], it follows that there is a structure of space with measured walls  $(X, W, B, \mu)$  and a point  $x_0 \in X$  such that  $\mu(W(gx_0 \mid g'x_0)) = \sqrt{\psi(g, g')} \geq \frac{1}{C}l(g^{-1}g')^\alpha$ , for all  $g, g' \in G$ . Next, from their Lemma 3.10, we deduce that for every  $p \geq 2$ , we have  $\alpha_p^*(G) \geq \alpha/p$ . Taking the supremum over  $\alpha$ , we obtain  $\alpha_p^*(G) \geq \alpha_2^*(G)/p$  for every  $p \geq 1$ .

Naor and Peres' Lemma 2.3 in [33] states that for every finitely generated group  $G$  and any  $p \geq 1$ , one actually has  $\alpha_p^*(G) \geq \alpha_2^*(G)$ . We generalize their result as follows.

**Theorem 4.2.** For every locally compact, compactly generated group  $G$ , equipped with the word length distance relative to a compact generating subset, and for any  $p \geq 1$ , we have  $\alpha_p^*(G) \geq \alpha_2^*(G)$ .

*Proof.* It is well known (see [29]) that any locally compact, compactly generated group  $G$  has a compact normal subgroup  $N$  such that  $G/N$  is separable. Let us define the length of any element  $\bar{a} \in G/N$  as  $\inf_{g \in \bar{a}} l(g)$ , where  $l$  is the word length metric on  $G$  with respect to a fixed compact generating subset  $K$ . Let  $\tilde{b} : G/N \rightarrow L_2$  be a 1-cocycle with respect to some affine isometric action  $\tilde{\pi}$  of  $G/N$  on some Hilbert space  $L_2$ . By separability of  $G/N$  and by continuity of  $\tilde{b}$ , we see that the image of  $\tilde{b}$  lies in a separable Hilbert subspace of  $L_2$ : indeed, take the closure of the subspace generated by the elements of  $\tilde{b}(C)$  where  $C \subset G/N$  is a countable dense subset. Denote this separable Hilbert space by  $l^2(\mathbb{Z}) \subset L_2$ . As  $l^2(\mathbb{Z})$  is generated by the image of  $\tilde{b}$ , we have that  $\tilde{\pi}(gN)(l^2(\mathbb{Z})) + \tilde{b}(gN) \subset l^2(\mathbb{Z})$  for all  $g \in G$ . In particular, one can for the purpose of calculating the equivariant Hilbert space compression of  $G/N$ , always consider actions on **separable** Hilbert spaces  $l^2(\mathbb{Z})$ . This fact turns out to be sufficient in order for the proof of Naor and Peres (Lemma 2.3 in [33]) to go through, i.e. we can conclude that  $\alpha_p^*(G/N) \geq \alpha_2^*(G/N)$  for all  $p \geq 1$ . One can check that  $\alpha_p^*(G) \geq \alpha_p^*(G/N)$ . Indeed, any affine isometric action  $\bar{\chi} = \bar{\pi} + \bar{b}$  of  $G/N$  on a Hilbert space gives rise to an affine isometric action  $\chi = \pi + b$  of  $G$  by precomposing with the quotient map  $G \rightarrow G/N$ . Moreover, if  $\bar{b}$  satisfies

$$\exists C, D \in \mathbb{R}^+, \forall g \in G : \|\bar{b}(\bar{g})\| \geq 1/C l'(\bar{g})^\alpha - D,$$

for some  $\alpha > 0$ , then we have

$$\|b(g)\| = \|\bar{b}(\bar{g})\| \geq 1/C l'(\bar{g})^\alpha - D \geq 1/C (l(g) - M)^\alpha - D,$$

where  $M \in \mathbb{N}$  is such that  $N \subset K^M$  and where  $g$  is such that  $|g| \geq M$ . We thus obtain  $\alpha_p^*(G) \geq \alpha_p^*(G/N) \geq \alpha_2^*(G/N)$ . The result then follows from the following lemma.  $\square$

**Lemma 4.3.** For any compact  $N \triangleleft G$ , we have  $\alpha_2^*(G/N) = \alpha_2^*(G)$ .

*Proof.* As above, one can check that  $\alpha_2^*(G) \geq \alpha_2^*(G/N)$ .

Conversely, to show that  $\alpha_2^*(G/N) \geq \alpha_2^*(G)$ , we start with any affine isometric action  $\chi = \pi + b$  of  $G$  on an  $L_2$ -space  $\mathcal{H}$ . Assume that

$$\exists C, D \in \mathbb{R}^+, \exists \alpha > 0, \forall g \in G : \|b(g)\| \geq 1/C l(g)^\alpha + D.$$

As  $N$  is compact, it has a fixed vector  $\xi$  (e.g. take  $\xi = \frac{1}{\mu(N)} \int_N b(q) d\mu$  where  $\mu$  is the Haar measure on  $G$ ). Define an action  $\tilde{\chi}$  of  $G$  on  $\mathcal{H}$  as follows:

$$\forall g \in G, v \in \mathcal{H} : \tilde{\chi}(g)(v) := \chi(g)(v + \xi) - \xi = \pi(g)(v) + \tilde{b}(g),$$

where  $\tilde{b}(g) = b(g) + \pi(g)(\xi) - \xi$ . Let  $\mathcal{H}^N \subset \mathcal{H}$  be the set of  $N$ -fixed vectors under  $\tilde{\chi}$ . Clearly,  $\tilde{b}$  vanishes on  $N$ , so  $\tilde{\chi}|_N = \pi|_N$  is a linear representation and  $\mathcal{H}^N$  is a Hilbert space. Note further that  $\|\tilde{b}(g) - b(g)\| \leq 2\|\xi\|$ , so the compression of  $b$  and  $\tilde{b}$  are the same.

Using that  $N$  is normal in  $G$ , one checks easily that  $\tilde{\chi}$  restricts to an action of  $G$  on  $\mathcal{H}^N$  which is constant on every  $N$ -coset. This induces naturally an affine isometric action  $\bar{\chi} = \bar{\pi} + \bar{b}$  of  $G/N$  on  $\mathcal{H}^N$ . Note that the compression of  $\bar{\chi}$  is larger than that of  $\chi$  because for all  $g \in G$ , we have

$$\begin{aligned} \|\bar{b}(\bar{g})\| &= \|\tilde{b}(g)\| \\ &\geq \|b(g)\| - 2\|\xi\| \\ &\geq 1/C l(g)^\alpha - D - 2\|\xi\| \\ &\geq 1/C l(\bar{g})^\alpha - D - 2\|\xi\|. \end{aligned}$$

We conclude that  $\alpha_2^*(G/N) \geq \alpha_2^*(G)$ .  $\square$

**Corollary 4.4.** *If  $G$  is an amenable locally compact hyperbolic group, then  $\alpha_p^*(G) = 1$  for every  $p \geq 1$ .*

*Proof.* As  $G$  is amenable, we have that  $\alpha_2^*(G) = \alpha_2(G)$  [19]. By Theorem 3.1, this shows that  $\alpha_2^*(G) = 1$ . Combining this with Theorem 4.2, we conclude that  $\alpha_p^*(G) = 1$  for every  $p \geq 1$ .  $\square$

**Corollary 4.5.** *For  $p \geq 2$ , we have  $\alpha_p^*(SU(n, 1)) = 1/2$ .*

*Proof.* Write  $G = SU(n, 1)$ . If we denote by  $d$  the hyperbolic distance and by  $x_0$  any point in complex hyperbolic space  $\mathbb{H}$ , then by combining a result by J. Faraut and K. Harzallah [20] with Theorem 2.1.1 in [15], we obtain that  $G$  admits an affine isometric action on a Hilbert space such that the associated 1-cocycle satisfies  $\|b(g)\|^2 = d(gx_0, x_0)$  for every  $g \in G$ . Consequently,  $\alpha_2^*(G) \geq 1/2$  as the map  $G \rightarrow \mathbb{H}, g \mapsto gx_0$  is a quasi-isometry. One then has that  $\alpha_p^*(G) \geq 1/2$  for all  $p \geq 1$ . Conversely, note that as  $G$  is of general type, it contains a quasi-isometrically embedded free non-abelian subgroup  $F_2$ . Using the fact that  $\alpha_p^*(F_2) = \max(1/2, 1/p)$ , we obtain  $\alpha_p^*(G) \leq \max(1/2, 1/p)$  as well. For  $p \geq 2$ , this implies  $\alpha_p^*(G) = 1/2$ .  $\square$

We can prove a stronger result for  $G = SO(n, 1)$ .

**Corollary 4.6.** *For every  $p \geq 1$ , we have  $\alpha_p^*(SO(n, 1)) = \max(1/2, 1/p)$ .*

*Proof.* It is known that  $SO(n, 1)$  acts on a space of walls. Precisely, denote real hyperbolic  $n$ -space by

$$X = \{x \in \mathbb{R}^{n+1} \mid x_n > 0, x_0^2 + x_1^2 + x_2^2 + \dots + x_{n-1}^2 - x_n^2 = -1\}.$$

Denote by  $\Omega$  the set of  $SO(n, 1)$ -translates of the half space

$$H = \{x \in \mathbb{R}^{n+1} \mid x_0 > 0, x_n > 0, x_0^2 + x_1^2 + x_2^2 + \dots - x_n^2 = -1\}.$$

Following [37], the stabilizer of  $H$  is  $SO(n-1, 1)$ , so  $\Omega$  is isomorphic to  $SO(n, 1)/SO(n-1, 1)$ . The groups  $SO(n, 1)$  and  $SO(n-1, 1)$  are both unimodular and so it follows that there is a nonzero positive  $SO(n, 1)$ -invariant measure  $\mu_\Omega$  on the quotient ([32], Chapter 3, p.140, Corollary 4). In particular, the natural linear action  $\lambda$  of  $SO(n, 1)$  on  $L_p(\Omega, \mu_\Omega)$  by “left translation” is unitary.

Given  $x$  in real hyperbolic space, denote  $\mathcal{A}_x$  the set of elements of  $\Omega$  that contain  $x$ . Corollary 2.5 in [37] states that there is a constant  $k > 0$  such that the hyperbolic distance  $d$  on  $X$  satisfies  $d(x, y) = k\mu_\Omega(\mathcal{A}_x \Delta \mathcal{A}_y)$  for all  $x$  and  $y$  in  $X$ .



We can now proceed as in [16] (see the proof of Proposition 2.8, (iii)  $\Rightarrow$  (iv')). Fix  $x_0 \in X$ . Denoting the characteristic function of  $\mathcal{A}_x$  by  $\chi_x$ , one checks easily that

$$b : SO(n, 1) \rightarrow L_p(\Omega, \mu_\Omega), g \mapsto \lambda(g)(\chi_{x_0}) - \chi_{x_0} = \chi_{gx_0} - \chi_{x_0},$$

is a 1-cocycle with respect to  $\lambda$ . Moreover,

$$\|b(g)\|_p = \mu_\Omega(\mathcal{A}_{gx_0} \Delta \mathcal{A}_{x_0})^{1/p} = \left(\frac{1}{k}\right)^{1/p} d(gx_0, x_0)^{1/p}.$$

As  $SO(n, 1) \rightarrow X, g \mapsto gx_0$  is a quasi-isometry, we conclude that  $\alpha_p^*(SO(n, 1)) \geq 1/p$  for every  $p \geq 1$ .

If we replace  $G = SU(n, 1)$  by  $G = SO(n, 1)$  in the proof of Corollary 4.5, we obtain  $1/2 \leq \alpha_p^*(G) \leq \max(1/2, 1/p)$ . Combining this with the above lower bound, we obtain  $\alpha_p^*(SO(n, 1)) = \max(1/2, 1/p)$ .  $\square$

**Remark 4.7.** *We are unable to generalize the above proof to calculate  $\alpha_p^*(SU(n, 1))$  for  $1 \leq p < 2$ . Indeed, Robertson's result that there is a constant  $k > 0$  such that the hyperbolic distance  $d$  on  $X$  satisfies  $d(x, y) = k\mu_\Omega(\mathcal{A}_x \Delta \mathcal{A}_y)$  for all  $x$  and  $y$  in  $X$ , is not valid in the complex hyperbolic case (see also [14], Corollary 6.28).*

## 5. PROPER AFFINE ISOMETRIC ACTIONS OF LOCALLY COMPACT HYPERBOLIC GROUPS ON $L_p$ -SPACES: A FIRST RESULT

With respect to the Haagerup property, there exist discrete hyperbolic groups on both ends of the spectrum. Similarly, in the non-discrete, locally compact case, we have locally compact hyperbolic groups which are Haagerup, e.g.  $SO(n, 1)$  and others which have property (T) and are thus far from being Haagerup, e.g.  $Sp(n, 1)$ . Although locally compact hyperbolic groups that are not Haagerup do *not* admit a proper affine isometric action on any  $L_2$ -space, we now show that they *do* admit a proper affine isometric action on an  $L_p$ -space for  $p$  sufficiently large. We will henceforth write  $L_p(G) = L_p(G, \mu)$  where  $\mu$  is the right Haar measure on  $G$ .

**Theorem 5.1.** *Let  $G$  be a non-amenable locally compact non-elementary hyperbolic group. Then  $G$  admits a metrically proper affine isometric action on a finite direct sum of copies of  $L_p(G)$  when  $p$  is sufficiently large. The linear part of the action is the direct sum of the natural translation actions on the  $L_p(G)$ . One can choose the corresponding 1-cocycle to have compression function  $\succeq t^{1/p}$ , so the compression of the associated 1-cocycle is greater than  $1/p$ .*

Our proof uses ideas from [4] and consists of three parts. First, we formally define a map  $c$  on  $G$  which satisfies the 1-cocycle relation with respect to the natural right translation action  $\rho$  on  $L^p(G, \mu)$ . Next, we show that  $c$  is also continuous: this must be verified as we only want to consider continuous  $G$ -actions. Finally, we show that  $c$  is also proper.

The proof that we present gives a concrete number  $N$ , in terms of properties of  $G$ , such that  $G$  admits a metrically proper affine isometric action on an  $L_p$ -space for all  $p \geq N$ . We refer the reader to Remark 5.5 for details. The discussion on page 5 of [31] indicates that the lower bound  $N$  obtained here is greater (i.e. less good) than the one we will obtain in Theorem 6.8. The key to improve the lower bound will be to strengthen Lemma 5.2 below. We give the proof of Theorem 5.1 because it is easier than that of Theorem 6.8, because most parts can be reused in Section 6, because it requires no additional preliminaries and because the obtained lower bound  $N$  is often easier to calculate than the Ahlfors regular conformal dimension.

### The proof of Theorem 5.1.

By Corollary 4.4, we can restrict our attention to the case where  $G$  is non-amenable, i.e. is of general type. Let  $X$  be a proper geodesic hyperbolic metric space admitting a proper cocompact  $G$ -action by isometries. From Proposition 5.10 in [12], we conclude that modulo a compact normal subgroup  $W < G$ , we have that  $G$  is totally disconnected or that it is the full isometry group (or its identity component) of a rank 1 symmetric space of noncompact type. So, in the former case, we can take  $X$  to be a Schreier graph (the set of vertices is  $\{\gamma V \mid \gamma \in G\}$  where  $V$  is an open compact subgroup of  $G$ ; see the remarks on the bottom of page 2 in [12] for more info). In the

latter case we can take  $X$  to be a rank 1 symmetric space of noncompact type. As  $W \triangleleft G$  is compact, we have  $\alpha_p^*(G) \geq \alpha_p^*(G/W)$  for every  $p \geq 1$ : indeed any affine isometric action of  $G/W$  on an  $L_p$ -space gives rise to one of  $G$  by precomposing with the projection map  $G \twoheadrightarrow G/W$ . We are thus allowed to henceforth assume  $G := G/W$ .

As before, let  $x_0 \in X$  and let  $d_a$  be a visual metric on  $\partial X$ . As the map  $G \rightarrow X, g \mapsto gx_0$  is a quasi-isometry, we can choose constants  $A > 0, B \geq 0$  such that

$$(3) \quad \frac{1}{A}d_G(g_1, g_2) - B \leq d_X(g_1x_0, g_2x_0) \leq Ad_G(g_1, g_2) + B.$$

Choose a Lipschitz function  $u$  on  $\partial X$  (for concreteness, let us choose a base point  $\xi_0 \in \partial X$  and set  $u : \xi \mapsto d_a(\xi_0, \xi)$ ). Denoting the right Haar measure on  $G$  by  $\mu$ , we define a function  $f_u : G \rightarrow \mathbb{R}$  as follows: set  $f_u : g \mapsto u(\xi_g)$  where  $\xi_g$  is any element in the boundary of  $X$  corresponding to a geodesic ray starting in  $x_0$  through  $gx_0$ . In the totally disconnected case, the stabilizer of  $x_0$  contains an open subset and so  $f_u$  is continuous. In the rank 1 case,  $f_u$  is continuous outside of the stabilizer of  $x_0$ , which is a measure 0 set  $S$ . Hence, in both cases,  $f_u$  is measurable.

Consider the standard right translation action  $\rho$  on  $L_p(G, \mu)$ , i.e. for every  $g \in G$  and  $f \in L_p(G, \mu)$  we set  $\rho(g)(f) : G \rightarrow \mathbb{R}, \gamma \mapsto f(\gamma g)$ . We formally define a 1-cocycle with respect to  $\rho$  as follows:

$$c := g \in G \mapsto \rho(g)(f_u) - f_u.$$

We first show that  $c(g) \in L_p(G, \mu)$  for every  $g \in G$  and  $p$  sufficiently large.

**Lemma 5.2.** *For  $p$  sufficiently large, we have  $\forall g \in G : c(g) \in L_p(G, \mu)$ .*

*Proof.* Let  $a, C \in \mathbb{R}$  denote constants as in Definition 2.2. Let  $K \subset G$  be a compact neighbourhood of 1 that generates  $G$ . Consider the cover  $(K\gamma_i)_{\gamma_i \in K^2}$  of  $K^2$ . Because of compactness, we can derive a finite subcover of  $K^2$ , say  $K\gamma_1, K\gamma_2, \dots, K\gamma_n$ . Note that for any element  $k_1k_2k_3 \in K^3$ , we have

$$k_1k_2k_3 = k_1(k_2k_3) = k_1k_4\gamma_i = (k_1k_4)\gamma_i = k_5\gamma_j\gamma_i,$$

for certain  $k_4, k_5 \in K$  and  $i, j \in \{1, 2, \dots, n\}$ . In particular,  $K^3$  is covered by the  $(K\gamma_i\gamma_j)_{i,j=1,2,\dots,n}$ . Continuing in this fashion, it is easy to check that  $K^m$  is covered by  $n^{m-1}$  right translates of  $K$ .

Fixing  $g \in G$  and neglecting the measure 0 set  $S$  mentioned earlier, we write

$$\begin{aligned} \|c(g)\|_p^p &= \int_G |f_u(\gamma g) - f_u(\gamma)|^p d\mu(\gamma) \\ &= \sum_{i=1}^{\infty} \int_{K^i \setminus K^{i-1}} |f_u(\gamma g) - f_u(\gamma)|^p d\mu \\ &\leq \sum_{i=1}^{\infty} \max_{\gamma \in K^i \setminus K^{i-1}} (|f_u(\gamma g) - f_u(\gamma)|^p) \mu(K^i \setminus K^{i-1}) \\ &\leq \sum_{i=1}^{\infty} \max_{\gamma \in K^i \setminus K^{i-1}} (|f_u(\gamma g) - f_u(\gamma)|^p) n^{i-1} \mu(K). \end{aligned}$$

By definition,

$$\begin{aligned} |f_u(\gamma g) - f_u(\gamma)| &= |u(\xi_{\gamma g}) - u(\xi_{\gamma})| \\ &\leq d_a(\xi_{\gamma g}, \xi_{\gamma}) \\ &\leq Ca^{-d(1, y_0)}, \end{aligned}$$

where  $y_0 \in (\xi_{\gamma g}, \xi_{\gamma}) \subset X$  lies at minimal distance from  $x_0$ . In order to bound  $|f_u(\gamma g) - f_u(\gamma)|$ , we look for a lower bound on  $d(x_0, y_0)$ .

For  $h \in \{\gamma g, \gamma\}$ , we let  $r_h$  be a geodesic ray in the class of  $\xi_h$  which starts in  $x_0$  and goes through  $hx_0$ . If  $x_h^i$  is the point on  $(hx_0, \xi_h)$  at distance  $i$  from  $hx_0$ , then a bi-infinite geodesic  $(\xi_{\gamma g}, \xi_{\gamma})$  can be obtained as the limit of geodesics  $[x_{\gamma}^i, x_{\gamma g}^i]$  as  $i \rightarrow \infty$ . These geodesics do not pass through the ball of radius

$$\min(d(x_0, \gamma gx_0), d(x_0, \gamma x_0)) - d(\gamma gx_0, \gamma x_0)/2 = \min(d(x_0, \gamma gx_0), d(x_0, \gamma x_0)) - d(gx_0, x_0)/2$$

around  $x_0$ . As  $d(x_0, \gamma g x_0) \geq d(x_0, \gamma x_0) - d(x_0, g x_0)$ , the geodesics do not pass through the ball of radius  $d(x_0, \gamma x_0) - 3/2 d(x_0, g x_0)$  around  $x_0$ . Consequently,

$$|f_u(\gamma g) - f_u(\gamma)| \leq C a^{-d(1, y_0)} \leq C a^{-[d(x_0, \gamma x_0) - 3/2 d(x_0, g x_0)]} := \bar{C} a^{-d(x_0, \gamma x_0)},$$

where we set  $\bar{C} = C a^{3/2 d(x_0, g x_0)}$ . As  $d(x_0, \gamma x_0) \geq \frac{1}{A} |\gamma| - B$ , we obtain that

$$|f_u(\gamma g) - f_u(\gamma)| \leq \tilde{C} a^{-\frac{|\gamma|}{A}},$$

where  $\tilde{C} = \bar{C} a^B$ . We thus have

$$(4) \quad \|c(g)\|_p^p \leq \tilde{C} \mu(K) \sum_{i=1}^{\infty} n^{i-1} a^{-\frac{p}{A}(i-1)} = \tilde{C} \mu(K) \sum_{i=1}^{\infty} (n a^{-\frac{p}{A}})^{i-1}.$$

Taking  $p$  such that  $n a^{-\frac{p}{A}} < 1$ , we get that  $\|c(g)\|_p < \infty$ , hence that  $c(g) \in L_p(G, \mu)$  for every  $g \in G$ .  $\square$

Next, we show that  $c$  is continuous.

**Lemma 5.3.** *Assume that  $p$  is such that  $c(g) \in L_p(G, \mu)$  for every  $g \in G$ . Then, the map  $c : G \rightarrow L_p(G, \mu)$  is continuous.*

*Proof.* As  $\|c(g) - c(h)\| = \|c(h^{-1}g) - c(1)\|$ , it suffices to show that  $c$  is continuous in  $1 \in G$ . So, fixing  $\epsilon$ , we need to find a neighbourhood  $1 \in V \subset G$  such that  $\|c(v)\| \leq \epsilon$  for every  $v \in V$ . In the case where  $G$  is totally disconnected, this is trivial: just take  $V$  inside the stabilizer of  $x_0$ . Let us thus focus on the rank 1 case. By Equation (4) in the previous lemma, and the fact that this infinite sum must converge, we can find an open  $1 \in V_1 \subset G$  and  $I$  large enough such that

- (1)  $\tilde{C} \mu(K) \sum_{i=I}^{\infty} (n a^{-\frac{p}{A}})^{i-1} \leq \epsilon/2$  for every  $v \in V_1$  (note that  $\tilde{C}$  depends on  $v \in V_1$ ),
- (2)  $K^{I-1} \supset \text{Stab}(x_0) V_1$ .

The problem is now reduced to choosing  $V \subset V_1$  small enough such that one can bound the norm of  $c(v)$ ,  $v \in V$  restricted to  $K^{I-1}$ . Take first  $V \subset V_1$  with  $V = V^{-1}$  such that  $\mu(\text{stab}(x_0)V) < \frac{\epsilon}{4 \text{diam}(\partial X)}$ . One can then bound the norm of  $c(v)$  restricted to  $K^{I-1}$  by  $\epsilon/4$  plus the norm of  $c(v)$  restricted to  $K^{I-1} \setminus \text{stab}(x_0)V$ . By decreasing  $V$  further if necessary, we can assure that the distance between  $\gamma v$  and  $\gamma$  becomes arbitrarily small. Moreover, for  $\gamma \in K^{I-1} \setminus \text{stab}(x_0)V$ , nor  $\gamma$ , nor  $\gamma v$  lies in  $\text{stab}(x_0)$ , so by continuity of  $f_u$  we have  $|f_u(\gamma v) - f_u(\gamma)|$  arbitrarily small. As continuity on a compact set implies uniform continuity, this leads to the conclusion.  $\square$

Finally, let us show that  $c$  is also proper.

**Lemma 5.4.** *The 1-cocycle  $c$  is proper.*

*Proof.* Choose  $g \in G$  and denote the length of  $g$  by  $m$ . Find elements  $k_1, k_2, \dots, k_m \in K$  whose product equals  $g$ . Define  $\gamma_i = (k_1 k_2 \dots k_{3i})^{-1}$  for  $i = 1, 2, \dots, \lfloor m/3 \rfloor$ , where  $\lfloor m/3 \rfloor$  is the largest integer smaller or equal to  $m/3$ . Write  $A = \cup K \gamma_i = \sqcup K \gamma_i$ . We obtain

$$\begin{aligned} \|c(g)\|_p^p &= \int_G |f_u(\gamma g) - f_u(\gamma)|^p d\mu(\gamma) \\ &\geq \int_A |f_u(\gamma g) - f_u(\gamma)|^p d\mu \\ &\geq \sum_{i=1}^{\lfloor m/3 \rfloor} \inf_{\gamma \in K \gamma_i} |f_u(\gamma g) - f_u(\gamma)|^p \mu(K) \\ &= \sum_{i=1}^{\lfloor m/3 \rfloor} \inf_{\gamma \in K \gamma_i} |u(\xi_{\gamma g}) - u(\xi_{\gamma})|^p \mu(K) \end{aligned}$$

For each  $k \in K$ , take the shortest path  $k \gamma_i g, k \gamma_i g k_m^{-1}, k \gamma_i g k_m^{-1} k_{m-1}^{-1}, \dots, k \gamma_i$ . It is clear that this path passes through  $k$  and hence through  $K$ . As  $G \rightarrow X, \gamma \mapsto \gamma x_0$  is a quasi-isometry and as  $X$  is hyperbolic, we conclude that the geodesics  $[k \gamma_i g x_0, k \gamma_i x_0]$  pass through the ball of radius  $R$  and centre  $x_0$  for some  $R > 0$  independent of  $g$ . Using a hyperbolicity argument, we see that there

exists  $k_0 \in \mathbb{N}$  and  $C > 0$  such that for  $g$  large enough and for  $i \in \{k_0, k_0 + 1, \dots, \lfloor m/3 \rfloor - k_0\}$ , the points  $\xi_{k\gamma_i g} = \xi_{\gamma g}$  and  $\xi_{k\gamma_i} = \xi\gamma$  are  $C$ -diametrically opposite, i.e. that  $(\xi_{\gamma g}, \xi_{\gamma})$  passes through the ball  $B(x_0, C)$ .

If we can find a constant  $C'$  and choose  $u$  such that  $C$ -diametrically opposite points  $\xi, \eta \in \partial G$  satisfy  $|u(\xi) - u(\eta)| \geq C' / (\mu(K))^{1/p}$ , then we can conclude that when  $|g|$  is sufficiently large:

$$\|c(g)\|_p \geq ((|g| - 2)/3 - 2k_0)^{1/p} C',$$

i.e. that  $\alpha_p^*(G) \geq \frac{1}{p}$ .

The map  $u : \partial G \rightarrow \mathbb{R}, \xi \mapsto d_a(\xi, \xi_0)$  that we have chosen does unfortunately not satisfy the above condition. We may however proceed as follows. As we are using a visual metric on  $\partial G$ , we obtain a constant  $\tilde{C}$  such that  $d_a(\xi, \eta) \geq \tilde{C}$  for any  $C$ -diametrically opposite points  $\xi, \eta \in \partial G$ . Now cover  $\partial G$  with open balls of radius  $\overline{C} := \tilde{C}/3$  and derive a finite subcover  $B(\xi_1, \overline{C}), B(\xi_2, \overline{C}), \dots, B(\xi_s, \overline{C})$ . For each  $i$ , define a Lipschitz map  $u_i : \partial G \rightarrow \mathbb{R}, \xi \mapsto d_a(\xi, \xi_i)$ . Given  $C$ -diametrically opposite points  $\chi, \eta \in \partial G$ , choose  $\xi_j$  such that  $d_a(\chi, \xi_j) \leq \overline{C}$ . As  $d_a(\chi, \eta) \geq 3\overline{C}$ , we obtain by the triangle inequality that  $|u_i(\chi) - u_i(\eta)| \geq \overline{C}$ . Said differently, there is a constant  $\overline{C}$  such that for any  $C$ -diametrically opposite points  $\chi, \eta \in \partial G$  there is some  $u_i$  ( $i = 1, 2, \dots, s$ ) such that  $|u_i(\chi) - u_i(\eta)| \geq \overline{C}$ .

One can now conclude by reasoning as above where we consider the diagonal action of  $G$  on  $L_p(\sqcup_{i=1}^s G)$  instead of  $L_p(G)$  and the cocycle  $c = (c_1, c_2, \dots, c_s)$  where  $c_i$  is associated to  $u_i$  as before.  $\square$

**Remark 5.5.** Assume that  $d_a$  is a visual metric on  $\partial X$ , where  $X$  is a proper hyperbolic geodesic metric space that admits a proper, cocompact isometric  $G$ -action. Take  $A > 0, B \geq 0$  such that  $G \rightarrow X, g \mapsto gx_0$  is an  $(A, B)$ -quasi-isometry. Take a compact generating subset of  $G$  and let  $n \in \mathbb{N}$  be such that  $K^2$  is contained in a union of  $n$  right translates of  $K$ . Then the proof of Lemma 5.2 shows that  $p$  is sufficiently large whenever  $p > A \frac{\ln(n)}{\ln(a)}$ .

## 6. PROPER AFFINE ISOMETRIC ACTIONS OF LOCALLY COMPACT HYPERBOLIC GROUPS ON $L_p$ -SPACES: THE IMPROVED RESULT

In this section we improve the bound given in Remark 5.5. Our new bound will be the *Ahlfors regular conformal dimension* of the hyperbolic boundary of  $G$ , as introduced in [6].

### 6.1. Preliminaries.

**6.1.1. Ahlfors regular conformal dimension.** Apart from visual metrics on the boundary of a hyperbolic group, one can also consider the class of Ahlfors regular metrics. The metric  $d$  on a compact metric space  $(Z, d)$  is called **Ahlfors regular** if its Hausdorff dimension  $Q$  and its  $Q$ -Hausdorff measure  $\nu$  satisfy

$$\frac{1}{C} r^Q \leq \nu(B(r)) \leq C r^Q,$$

for every ball  $B(r) \subset (Z, d)$  with  $r \leq \text{diam}(Z, d)$  and where  $C > 0$  is a constant that is independent of  $r$ . Any compact metric space  $(Z, d)$  with  $d$  Ahlfors regular satisfies the following properties.

**Definition 6.1.** A compact metric space  $(Z, d)$  is **uniformly perfect** if there exists a constant  $C > 1$  such that for any ball  $B(x, r) \subset Z$  with  $0 < r \leq \text{diam}(Z)$ , we have  $B(x, r) \setminus B(x, r/C) \neq \emptyset$ . A compact metric space  $(Z, d)$  is called **doubling** if there exists a constant  $C \geq 1$  such that for each  $r > 0$ , each ball of radius  $r$  can be covered by at most  $C$  balls of radius  $r/2$ . Related to this, a measure  $\nu$  on  $(Z, d)$  is called **doubling** if open balls all have finite measure and if there exists a constant  $C$  such that for each  $r > 0$  and  $z \in Z$ , we have  $\nu(B(z, 2r)) \leq C\nu(B(z, r))$ . It is easy to check that the  $Q$ -Hausdorff measure  $\nu$  of an Ahlfors regular (hence doubling) metric space of Hausdorff dimension  $Q$ , is doubling.

Corollary 14.15 in [25] shows that conversely, if  $(Z, d)$  satisfies these properties (i.e. is uniformly perfect and has a doubling measure), then it can be equipped with an Ahlfors regular metric *quasi-symmetric* to  $d$ .

**Definition 6.2.** Two metrics  $d, d'$  on a compact metric space  $(Z, d)$  are **quasi-symmetric** if they induce the same topology and if there exists a homeomorphism  $\eta : (0, \infty) \rightarrow (0, \infty)$  such that for any  $x, y, z \in Z$  and  $t \in [0, \infty)$ , we have

$$\frac{d(x, z)}{d(y, z)} \leq t \Rightarrow \frac{d'(x, z)}{d'(y, z)} \leq \eta(t).$$

We note that the previous implication also implies

$$\frac{d'(x, z)}{d'(y, z)} \leq t \Rightarrow \frac{d(x, z)}{d(y, z)} \leq \psi(t) := 1/\eta^{-1}(t^{-1}).$$

As an example of an Ahlfors regular metric, we can take any visual metric on the boundary of a discrete hyperbolic group. More generally, given a locally compact hyperbolic group  $G$ , it acts properly cocompactly isometrically on a proper hyperbolic geodesic metric space  $X$ . The boundary of  $G$  coincides with that of  $X$  by definition. One can now show (by verifying the conditions in [5], bottom of page 2) that the boundary of a locally compact hyperbolic group, equipped with a visual metric, is uniformly perfect and doubling. As the boundary is also a complete metric space, it carries a doubling measure (Theorem 13.3 in [25]). In particular, any visual metric is quasi-symmetric to an Ahlfors regular metric.

It is often convenient to work with different characterizations of quasi-symmetric metrics. We state the following.

**Proposition 6.3.** Let  $(Z, d)$  be a compact, uniformly perfect metric space and assume that  $d'$  is a metric on  $Z$  that induces the same topology as  $d$ . Consider the following three conditions.

- (1)  $d$  and  $d'$  are quasi-symmetric.
- (2)  $d$  is Hölder equivalent to  $d'$ , i.e. there exists  $\alpha, \beta, C > 0$  such that for all  $x, y \in Z$ , we have

$$\frac{1}{C} d'(x, y)^\beta \leq d(x, y) \leq C d'(x, y)^\alpha.$$

- (3)  $d'$  is annulus semi-quasiconformal with respect to  $d$ . This means that if we define  $\bar{d}_d(x, r) = \sup_{y \in B_d(x, r)} d(x, y)$ , then for any  $\epsilon \in (0, 1)$ , there exists  $\delta \in (0, 1)$  such that a  $d'$ -annulus  $B_{d'}(x, r) \setminus B_{d'}(x, \epsilon r)$  is contained in a  $d$ -annulus  $B_d(x, (1 + \epsilon)\bar{d}_d(x, r)) \setminus B_d(x, \delta \bar{d}_d(x, r))$  for any  $\epsilon > 0$ .

Then,  $1 \rightarrow 2$  and  $1 \leftrightarrow 3$ .

*Proof.* The implication  $(1) \rightarrow (2)$  follows from Remark 1.1.b in [3]. A proof of the equivalence  $(1) \leftrightarrow (3)$  can be found in [30], beginning of part 2.  $\square$

**Definition 6.4.** Let  $(Z, d)$  be a compact, uniformly perfect, doubling metric space. One defines the Ahlfors regular conformal gauge  $\mathcal{J}_{AR}(Z, d)$  of  $(Z, d)$  as

$$\mathcal{J}_{AR}(Z, d) = \{d' \mid d' \text{ is an Ahlfors regular metric on } Z \text{ quasi-symmetric to } d\}.$$

Visual metrics on the boundary of a proper hyperbolic metric space are always quasi-symmetric. We may thus abbreviate  $\mathcal{J}_{AR}(Z, d)$  by  $\mathcal{J}_{AR}(\partial G)$  if  $(Z, d)$  is the boundary of a locally compact hyperbolic group  $G$ , equipped with a visual metric.

**Definition 6.5.** Let  $(Z, d)$  be a compact, uniformly perfect, doubling metric space. The Ahlfors regular conformal dimension,  $\text{Confdim}(Z, d)$ , of  $(Z, d)$  is defined as

$$\text{Confdim}(Z, d) = \inf\{\text{Hausdim}(Z, d') \mid d' \in \mathcal{J}_{AR}(Z, d)\},$$

where  $\text{Hausdim}(Z, d')$  denotes the Hausdorff dimension of  $(Z, d')$ . Again, if  $(Z, d)$  is the hyperbolic boundary of a hyperbolic group  $G$ , equipped with a visual metric, then we write  $\text{Confdim}(Z, d) = \text{Confdim}(\partial G)$ .

6.1.2. *Shadows.* Given a ball in  $X$ , say of radius  $R$  and centre  $x$ , we define its shadow by

$$\mathcal{S}(x, R) = \{\xi \in \partial X \mid \exists \text{ a geodesic ray } r \in \xi \text{ starting in } x_0 \text{ such that } r \cap B(x, R) \neq \emptyset\}.$$

For later use, we require the following observation.

**Lemma 6.6.** *Let  $x_0 \in X$  be some base-point and fix some  $c \in \partial G$ . Assume that  $A \subset \partial G$  is an open neighbourhood of  $c$ . Then, any geodesic ray in the class of  $c$  emanating from  $x_0$  contains a point  $x$  such that  $\mathcal{S}(x, 2R) \subset A$ .*

*Proof.* Take  $r > 0$  such that the open ball  $B(c, r)$  (with respect to a visual metric  $d_{a,C}$ ) is contained in  $A$ . Next, take  $x$  on a geodesic ray in the class of  $c$  that emanates from  $x_0$  such that  $d(x, x_0) > \ln(\frac{Ca^{3R}}{r})/\ln(a)$ . Then, given  $\xi \in \mathcal{S}(x, 2R)$ , we have that there is a geodesic ray in the class of  $\xi$  which starts in  $x_0$  and passes through a point  $y \in B(x, 2R)$ . We can construct a bi-infinite geodesic  $(\xi, c)$  by taking the limit of geodesics between points on  $(x_0, \xi)$  and  $(x_0, c)$  that tend to infinity. Such geodesics stay outside the ball of radius  $d(x_0, x) - 3R$  around  $x_0$ . So, in particular,  $d_{a,C}(\xi, c) \leq Ca^{3R-d(x_0, x)}$ . As  $d(x, x_0) > \ln(\frac{Ca^{3R}}{r})/\ln(a)$ , we obtain  $d_{a,C}(\xi, c) < r$  as desired.  $\square$

We will also use the following lemma.

**Lemma 6.7.** *Let  $G$  be a locally compact hyperbolic group and let  $X$  be a proper hyperbolic geodesic metric space that admits a proper, cocompact action of  $G$  by isometries. Take  $\delta$  such that  $X$  is  $\delta$ -hyperbolic and choose  $x_0 \in X$  and  $R > \max(1, 20\delta)$  such that for every  $g \in G : \mathcal{S}(gx_0, R) \neq \emptyset$ .*

*Let  $d$  be a metric in  $\mathcal{J}_{AR}(\partial G)$ . There exists a constant  $D > 0$  such that for all  $g \in G$  there exists  $r > 0$  and  $\xi \in \partial G$  with*

$$B_d(\xi, r) \subset \mathcal{S}(gx_0, 2R) \subset B_d(\xi, Dr).$$

*Proof.* Let us first assume that  $d = d_a$  is a visual metric on  $\partial G = \partial X$ . Choose any point  $g \in G, \xi \in \mathcal{S}(gx_0, R)$  and let  $C, a > 1$  be as in Definition 2.2. We claim that

$$B(\xi, 1/(Ca^{d(x_0, gx_0)+R+20\delta})) \subset \mathcal{S}(gx_0, 2R).$$

To verify this, choose  $\beta \in B(\xi, 1/(Ca^{d(x_0, gx_0)+R+20\delta}))$ . Let  $\tilde{\xi} \in [x_0, \xi] \cap B(gx_0, R)$  and let  $y$  be the point on the bi-infinite geodesic  $(\xi, \beta)$  closest to  $x_0$ . Then,  $d(x_0, y) > d(x_0, gx_0) + R + 20\delta$ . So,  $\tilde{\xi}$  lies more than  $20\delta$  away from the bi-infinite geodesic  $(\xi, \beta)$ . As ideal geodesic triangles are  $20\delta$ -thin,  $\tilde{\xi}$  lies less than a distance  $20\delta$  from a point  $P$  on the geodesic  $(x_0, \beta)$ . We thus have  $\beta \in \mathcal{S}(gx_0, R + 20\delta) \subset \mathcal{S}(gx_0, 2R)$  as desired.

Conversely, we claim that

$$\mathcal{S}(gx_0, 2R) \subset B(\xi, C/a^{d(x_0, gx_0)-4R}),$$

for every  $\xi \in \mathcal{S}(gx_0, 2R)$ . To show this, assume  $\xi, \beta \in \mathcal{S}(gx_0, 2R)$  and take points  $\tilde{\xi} \in (x_0, \xi) \cap B(gx_0, 2R)$  and  $\tilde{\beta} \in (x_0, \beta) \cap B(gx_0, 2R)$ . For every  $i \in \mathbb{N}$ , denote  $\tilde{\xi}_i$  ( $\tilde{\beta}_i$ ) the unique point in  $[\tilde{\xi}, \xi]$  ( $[\tilde{\beta}, \beta]$  respectively) at distance  $i$  from  $\tilde{\xi}$  ( $\tilde{\beta}$  respectively). As  $d(\tilde{\xi}, \tilde{\beta}) \leq 4R$ , one sees that the geodesics  $(\tilde{\xi}_i, \tilde{\beta}_i)$  stay outside  $B(x_0, d(x_0, gx_0) - 4R)$ . Moreover, these geodesics converge to the bi-infinite geodesic  $(\xi, \beta)$ . So, if  $y \in (\xi, \beta)$  is the point closest to  $x_0$ , then  $d(x_0, y) > d(x_0, gx_0) - 4R$  and

$$d_a(\xi, \beta) \leq Ca^{-d(x_0, y)} < C/a^{d(x_0, gx_0)-4R}.$$

We thus obtain that for every  $g \in G$ , there exists  $r > 0$  such that for all  $\xi \in \mathcal{S}(gx_0, R)$  we have

$$B(\xi, r) \subset \mathcal{S}(gx_0, 2R) \subset B(\xi, Dr),$$

where

$$D = C^2 a^{5R+20\delta}.$$

Now, if  $d$  is any metric in  $\mathcal{J}_{AR}(\partial G)$ , then it must be quasi-symmetric to  $d_a$ . By Proposition 6.3,  $d$  is then annulus semi-quasiconformal to  $d_a$ , hence the result follows.  $\square$

**6.2. A better bound.** In this paragraph, we prove the following result.

**Theorem 6.8.** *Let  $G$  be a non-amenable locally compact non-elementary hyperbolic group and denote the Ahlfors regular conformal dimension of  $\partial G$  by  $Q$ . Then for each  $p > Q$ , there is a metrically proper affine isometric action of  $G$  on a finite  $l_p$ -direct sum of copies of  $L_p(G)$ . The linear part of the action is the finite direct sum of the natural translation actions. One can choose the corresponding 1-cocycle to have compression function  $\succeq t^{1/p}$ . In particular, for every  $p > Q$ , we have  $\alpha_p^*(G) \geq 1/p$ .*

*Proof.* This proof uses similar notations as in Section 5. So, let  $(X, d_X)$  be a proper hyperbolic geodesic metric space that admits a proper cocompact isometric  $G$ -action. Let  $K$  be a compact generating neighbourhood of  $1 \in G$ . Take  $\delta$  such that  $X$  is  $\delta$ -hyperbolic and choose  $x_0 \in X$  and  $R > \max(1, 20\delta)$ . Take  $A > 0, B \geq 0$  such that  $G \rightarrow X, \gamma \mapsto \gamma x_0$  is an  $(A, B)$ -quasi-isometry. An open ball in  $X$  with centre  $x$  and radius  $r$  will be denoted by  $B_X(x, r)$ . Let  $d \in \mathcal{I}_{AR}(\partial G)$  and denote the Hausdorff measure and Hausdorff dimension of  $(\partial G, d) = (\partial X, d)$  by  $\nu$  and  $Q$  respectively. Choose a base point  $\xi_0 \in \partial G$  and define a Lipschitz function  $u : \partial G \rightarrow \mathbb{R}, \xi \mapsto d(\xi_0, \xi)$ . We define a function  $f_u : G \rightarrow \mathbb{R}$  as follows: set  $f_u : g \mapsto u(\xi_g)$  where  $\xi_g$  is any element in the boundary of  $X$  corresponding to a geodesic ray starting in  $x_0$  through  $gx_0$ . Denote the right Haar measure on  $G$  by  $\mu$  and consider the standard translation action  $\rho$  on  $L_p(G, \mu)$ , i.e. for every  $g \in G$ , we set  $\rho(g)(f) : G \rightarrow \mathbb{R}, \gamma \mapsto f(\gamma g)$ . Let us formally define the following 1-cocycle with respect to  $\rho$ :

$$c := g \in G \mapsto \rho(g)(f_u) - f_u.$$

We now show that  $c(g) \in L_p(G, \mu)$  for all  $g \in G$  and  $p > Q$ . Once we have done this, we get the properness and continuity of  $c$  for granted by the Lemmas 5.3 and 5.4. Our proof uses ideas from Proposition 2.3 in [4].

Choose  $g \in G$  and write  $g = k_1 k_2 \dots k_m$  where  $k_i \in K$  and  $m = |g|$  is the length of  $G$ . Writing  $g_i = k_1 k_2 \dots k_i$ , we obtain a constant  $C(m)$  such that

$$\begin{aligned} \|c(g)\|_p^p &= \int_G |f_u(\gamma g) - f_u(\gamma)|^p d\mu(\gamma) \\ &= \int_G \left| \sum_{i=1}^m f_u(\gamma g_i) - f_u(\gamma g_{i-1}) \right|^p d\mu \\ &\leq C(m) \sum_{i=1}^m \int_G |f_u(\gamma g_i) - f_u(\gamma g_{i-1})|^p d\mu \\ &= C(m) \sum_{i=1}^m \|c(k_i)\|_p^p. \end{aligned}$$

It thus suffices to consider the case  $g = k \in K$ .

For each  $\gamma \in G$ , let us denote the minimal radius of a  $d$ -ball in  $\partial G$  containing  $\mathcal{S}(\gamma x_0, 2R)$  by  $r(\gamma)$ . Note that  $\mathcal{S}(\gamma x_0, 2R) \cap \mathcal{S}(\gamma k x_0, 2R) \neq \emptyset$ , so  $d(\xi_{\gamma k}, \xi_\gamma) \leq 2(r(\gamma k) + r(\gamma))$ . Using the fact that  $u$  is Lipschitz and that  $\mu$  is right-invariant, we obtain a constant  $C_1 > 1$  such that

$$\|c(k)\|_p^p \leq C_1 \int_G r(\gamma)^p d\mu.$$

By Lemma 6.7, there is  $i \in \mathbb{N}$  such that for each  $\gamma \in G$ , there is  $\xi \in \partial G$ , such that  $B(\xi, r(\gamma)/2^i) \subset \mathcal{S}(\gamma x_0, 2R) \subset B(\xi, r(\gamma))$ . So, by Ahlfors regularity (and the fact that Ahlfors regular measures are doubling), we see that for some  $\xi \in \partial G$ , we have

$$r(\gamma)^Q \leq C_2 \nu(B(\xi, r(\gamma))) = C_2 \nu(B(\xi, 2^i r(\gamma)/2^i)) \leq C_2 C_3^i \nu(B(\xi, r(\gamma)/2^i)) \leq C_2 C_3^i \nu(\mathcal{S}(\gamma x_0, 2R)),$$

where  $C_2$  and  $C_3$  are the constants appearing in the definition of Ahlfors regular metric and doubling measure respectively. Writing  $C_2 := C_1(C_2 C_3^i)^{p/Q}$ , we thus obtain

$$\|c(k)\|_p^p \leq C_2 \int_G (\nu(\mathcal{S}(\gamma x_0, 2R)))^{p/Q} d\mu$$

Write  $V = K^{A(8R+2\delta+1+B)}$ ,  $\epsilon_i = \frac{1}{2^i}$  ( $i \in \mathbb{N}$ ) and choose  $\gamma_0 \in G$  such that  $\nu(\mathcal{S}(\gamma_0 x_0, 2R)) \geq \sup_{\gamma \in G} \nu(\mathcal{S}(\gamma x_0, 2R)) - \epsilon_0$ . As a next step, choose  $\gamma_1 \in G \setminus (\gamma_0 V)$  such that  $\nu(\mathcal{S}(\gamma_1 x_0, 2R)) \geq \sup_{\gamma \in G \setminus \gamma_0 V} \nu(\mathcal{S}(\gamma x_0, 2R)) - \epsilon_1$  and continue in this manner. So, choose  $\gamma_2 \in G \setminus (\gamma_0 V \cup \gamma_1 V)$  such that  $\nu(\mathcal{S}(\gamma_2 x_0, 2R)) \geq \sup_{\gamma \in G \setminus (\gamma_0 V \cup \gamma_1 V)} \nu(\mathcal{S}(\gamma x_0, 2R)) - \epsilon_2, \dots$ . We claim that we so obtain a countable cover  $\gamma_i V$  of  $G$ . To show this, assume by contradiction that  $\cup_{i \in \mathbb{N}} \gamma_i V \neq G$ . Using the proof of Lemma 6.7, we see that any shadow  $\mathcal{S}(y, 2R)$  with  $y \in X$  contains an open ball with radius say  $d_y > 0$ . Hence, by Ahlfors regularity, it has strictly positive measure. Given  $\tilde{\gamma} \in G \setminus \cup_{i \in \mathbb{N}} \gamma_i V$ , we thus find  $\tilde{\epsilon} > 0$  such that  $\nu(\mathcal{S}(\tilde{\gamma} x_0, 2R)) = \tilde{\epsilon}$ . By construction of the sequence  $(\gamma_i)_i$ , this thus means that  $\nu(\mathcal{S}(\gamma_i x_0, 2R)) > \tilde{\epsilon} - \epsilon_i > \tilde{\epsilon}/2$  for  $i$  sufficiently large. Again from the proof of Lemma 6.7 and the fact that  $d$  is Hölder equivalent with a visual metric, we can find a compact neighbourhood  $L \subset G$  of 1 such that  $\nu(\mathcal{S}(\gamma x_0, 2R)) < \tilde{\epsilon}/2$  whenever  $\gamma \notin L$ . In particular, infinitely many of the  $\gamma_i$  must lie in  $L$ . As the sets  $\gamma_j V$  are all disjoint, the set of such  $\gamma_i \in L$  does not contain a convergent subsequence and this contradicts compactness of  $L$ . We thus conclude that the  $(\gamma_i V)_{i \in \mathbb{N}}$  form a countable cover of  $G$  by left translates of  $V$  such that the word length distance  $d_V(\gamma_i, \gamma_j) \geq 2$  for all  $i \neq j \in \mathbb{N}$ .

Denote  $\mathcal{G} = \{\gamma_i \mid i \in \mathbb{N}\}$  and for each  $n \in \mathbb{N}$ , define

$$\mathcal{A}(n) = \{\gamma \in G \mid \gamma x_0 \in B_X(x_0, n) \setminus B_X(x_0, n-1)\}.$$

We can find constants  $D_1, C_3 > 0$  such that

$$\begin{aligned} \|c(k)\|_p^p &\leq C_2 \sum_{\gamma_i \in \mathcal{G}} (\nu(\mathcal{S}(\gamma_i x_0, 2R)) + \epsilon_i)^{p/Q} \mu(V) \\ &= C_2 \sum_{n \in \mathbb{N}} \sum_{\gamma_i \in \mathcal{G} \cap \mathcal{A}(n)} (\nu(\mathcal{S}(\gamma_i x_0, 2R)) + \epsilon_i)^{p/Q} \mu(V) \\ &\leq D_1 + C_3 \sum_{n \in \mathbb{N}} \sum_{\gamma_i \in \mathcal{G} \cap \mathcal{A}(n)} \nu(\mathcal{S}(\gamma_i x_0, 2R))^{p/Q} \mu(V) \\ &= D_1 + C_3 \sum_{n \in \mathbb{N}} \sum_{\gamma_i \in \mathcal{G} \cap \mathcal{A}(n)} \nu(\mathcal{S}(\gamma_i x_0, 2R))^{\frac{p-Q}{Q}} \nu(\mathcal{S}(\gamma_i x_0, 2R)) \mu(V). \end{aligned}$$

We start by bounding the factor  $(\nu(\mathcal{S}(\gamma_i x_0, 2R)))^{p-Q/Q}$ . Recall that  $\mathcal{S}(\gamma_i x_0, 2R)$  is contained in a ball of radius  $r(\gamma)$ , so by Ahlfors regularity, there is a constant  $C_4 > 0$  such that  $\nu(\mathcal{S}(\gamma_i x_0, 2R)) \leq C_4 r(\gamma_i)^Q$ . If  $\tilde{d}$  is some visual metric on the boundary, then there are constants  $\tilde{C}_5, \tilde{b} > 1$  such that  $r(\gamma) \leq \tilde{C}_5 \tilde{b}^{-d_X(x_0, \gamma x_0)}$ , where  $r(\gamma)$  is the minimal radius of a  $\tilde{d}$ -ball in  $\partial X$  that contains  $\mathcal{S}(\gamma x_0, 2R)$ . As  $d \in \mathcal{J}_{\text{AR}}(\partial G)$ , it is Hölder equivalent to a visual metric. So, there are constants  $\overline{C}_5, r > 1$  such that  $r(\gamma) \leq \overline{C}_5 r(\gamma)^r$ . We then have

$$r(\gamma) \leq \overline{C}_5 r(\gamma)^r \leq \overline{C}_5 \tilde{C}_5^r (\tilde{b}^r)^{-d_X(x_0, \gamma x_0)} = C_5 b^{-d_X(x_0, \gamma x_0)},$$

where  $C_5 := \overline{C}_5 \tilde{C}_5^r$  and  $b := \tilde{b}^r$ . This implies

$$\nu(\mathcal{S}(\gamma_i x_0, 2R))^{(p-Q)/Q} \leq C_4^{(p-Q)/Q} (r(\gamma_i))^{p-Q} \leq C_4^{(p-Q)/Q} C_5^{p-Q} b^{-d_X(x_0, \gamma x_0)(p-Q)}.$$

Writing  $C_4 := C_3 C_4^{(p-Q)/Q} C_5^{p-Q}$ , we thus obtain

$$\|c(k)\|_p^p \leq D_1 + C_4 \mu(V) \sum_{n \in \mathbb{N}} b^{-(n-1)(p-Q)} \sum_{\gamma_i \in \mathcal{G} \cap \mathcal{A}(n)} \nu(\mathcal{S}(\gamma_i x_0, 2R)).$$

In order to bound the latter sum, we choose  $n \in \mathbb{N}$  and assume for a moment by contradiction that one can choose  $\tilde{g}_1, \tilde{g}_2 \in \mathcal{G} \cap \mathcal{A}(n)$  in such a way that  $\mathcal{S}(\tilde{g}_1 x_0, 2R) \cap \mathcal{S}(\tilde{g}_2 x_0, 2R) \neq \emptyset$ . Take then  $\xi$  in the intersection and let  $c_1, c_2$  be geodesic rays starting in  $x_0$  in the class of  $\xi$  such that  $c_1$  intersects  $B(\tilde{g}_1 x_0, 2R)$  and  $c_2$  intersects  $B(\tilde{g}_2 x_0, 2R)$ . Let  $P_1 \in c_1$  be the point closest to  $\tilde{g}_1 x_0$ . It is clear that  $d_X(P_1, \tilde{g}_1 x_0) \leq 2R$  and  $n-1-2R \leq d_X(P_1, x_0) \leq n+2R$  because  $\tilde{g}_1 \in \mathcal{A}(n)$ . Similarly, choosing  $P_2 \in c_2$  the point closest to  $\tilde{g}_2 x_0$ , we see that  $d_X(P_2, \tilde{g}_2 x_0) \leq 2R$  and  $n-1-2R \leq d_X(P_2, x_0) \leq n+2R$ . As  $c_1$  and  $c_2$  are both geodesic rays in the class of  $\xi$  that start in  $x_0$ , they lie at Hausdorff-distance less than  $\delta$  from each other. So,  $d_X(P_1, P_2) \leq 4R+1+2\delta$  and so  $d_X(\tilde{g}_1 x_0, \tilde{g}_2 x_0) \leq 8R+1+2\delta$ . We thus obtain  $d_K(\tilde{g}_1, \tilde{g}_2) \leq A(8R+1+2\delta+B)$  and so



$d_V(\tilde{g}_1, \tilde{g}_2) \leq 1$ , which leads to a contradiction as by construction  $d_V(\gamma_i, \gamma_j) \geq 2$  for all  $\gamma_i, \gamma_j \in \mathcal{G}$ . We can thus bound  $\sum_{\gamma_i \in \mathcal{G} \cap \mathcal{A}(n)} \nu(\mathcal{S}(\gamma_i x_0, 2R)) \leq \nu(\partial G)$  and obtain

$$\|c(k)\|_p^p \leq D_1 + C_4 \mu(V) \nu(\partial G) \sum_{n \in \mathbb{N}} b^{-(n-1)(p-Q)}.$$

Hence,  $c(g) \in L_p(G, \mu)$  for all  $g \in G$  and  $p > Q$ .

The proof that  $c$  is proper can then be copied word for word from the proof of Lemma 5.4.  $\square$

Using that the Ahlfors regular conformal dimension of the boundary of  $Sp(n, 1)$  and  $F_4^{-20}$  are known to be  $4n + 2$  and  $22$  respectively [36], we obtain the following corollary.

**Corollary 6.9.** *The group  $G = F_4^{-20}$  for  $p > 22$  and the groups  $G = Sp(n, 1)$  for  $p > 4n + 2$  admit proper affine isometric actions on a finite  $l_p$ -direct sum of copies of  $L_p(G)$ . Moreover, the linear part of the action is the direct sum of the natural actions by translation on the copies of  $L_p(G)$  and the corresponding 1-cocycle has compression at least  $1/p$ .*

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